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# Statistical physics of media processes: Mediaphysics

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**Abstract**

The processes of mass communications in complicated social or sociobiological systems such as marketing, economics, politics, animal populations, etc. as a subject for the special scientific subbranch—''mediaphysics''—are considered in its relation with sociophysics. A new statistical physics approach to analyze these phenomena is proposed. A keystone of the approach is an analysis of population distribution between two or many alternatives: brands, political affiliations, or opinions. Relative distances between a state of a ''person's mind'' and the alternatives are measures of propensity to buy (to affiliate, or to have a certain opinion). The distribution of population by those relative distances is time dependent and affected by external (economic, social, marketing, natural) and internal (influential propagation of opinions, ''word of mouth'', etc.) factors, considered as fields. Specifically, the interaction and opinion-influence field can be generalized to incorporate important elements of Ising-spin-based sociophysical models and kinetic-equation ones. The distributions were described by a Schrödinger-type equation in terms of Green's functions. The developed approach has been applied to a real mass-media efficiency problem for a large company and generally demonstrated very good results despite low initial correlations of factors and the target variable.
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## 1. Introduction

Processes of mass communications take place practically in all social and sociobiological systems [1,2]. In ''real life'' the mass communications are driving forces for many phenomena in marketing, economics, and politics. Traditionally, these phenomena are described by statistical approaches without a deep understanding and involving of the driving forces. The object of the present paper is a methodology of application of *statistical physics* to the complicated processes of mass communications. We called this kind of applications ''mediaphysics'', which is a part of sociophysics, studying processes of mass communications in social and sociobiological systems.

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To make it clear, let us consider two data sets that are typical for statistics: (a) 20 weekly observations of J. Smith's Coca Cola purchases, and (b) 20 weekly observations of Coca Cola sales in the USA. Traditional statistics will treat those sets identically. However, they are qualitatively different. In case (a) the level of purchase is explained by individual behavior of J. Smith, his habits, income, etc. In case (b) the same explanation is applied too (all individuals, buying soda in the USA, have motivations, similar to J. Smith's one); but on the top of that, there is a structure of these individuals by their distribution in readiness and willingness to buy soda in general and Coca Cola specifically. Effects of advertising activities, economic factors and opinion exchange (word of mouth) depend on this distribution. The distribution is changing over time and that is something very important which does not exist in case (a). The observed sales value for the USA is in fact a result of millions of individual actions, which appear mostly in an unobserved form. The observations (sales, etc.) are just the tip of the iceberg.

Therefore, apart from its own subject, the mediaphysics can be characterized by the fact that it has to operate with *real-life data of certain types* that are untypical for traditional statistics and usual for statistical physics. Traditionally, social and biological sciences used statistics (not statistical physics) to describe the above-mentioned phenomena (a) and (b), ignoring the deep differences between two data types, while using just the observed ones. We propose to analyze systems like (b) using developed techniques of statistical physics. It is an appropriate tool due to the deep similarities between behaviors of particles or molecules en masse (typical unobserved units), and processes described above. Indeed, using traditional-statistical physics we can analyze the observed values of gas pressure and temperature without a detailed information about each particle, but knowing their distribution, responses to different fields and interaction with each other.

The placement of mediaphysics into sociophysical realm [3–5] needs some comments. Since it is related with disputable questions on history and terminology of the crossroads of sociology, economy, statistics, mathematics and theoretical physics, but does not directly touch the subject matter of this paper, we put our historical and terminological views and comments into Appendix. Here let us just say that the introduced mediaphysics overlaps with sociophysics and statistics, but focuses on communications and thus belongs to the sociophysics field inside its universal and broad definitions (see Appendix). The mediaphysics orients to real-life data, which currently is not typical for sociophysics simulations. It deals with both observed and unobserved data unlike traditional-statistical approaches. Plus, mediaphysics is associated with two meanings of the term "media", both of which are relevant to the approach: media as an environment in which mass processes of communications are taking place; and in a form of "mass-media", as an advertising (or other messages) spreading through mass-communication channels, which itself is a very important topic.

The article is organized as follows. Section 2 contains a discussion of the problem and defines new terms (it introduces important concepts used thereafter, especially one of mindset's space and motion in there); in Section 3 we describe the main formalism, based on Green's functions and Schrödinger-type equation; Section 4 demonstrates how this approach is related with two important modern statistical techniques; then in Section 5 we discuss model implementation to the real data; and Section 6 presents conclusion remarks.

## 2. Distributions in a space of persons' mindsets

In the modern world people are subjected to hundreds of activities intended to attract customers, voters, or followers. It creates a strong competitive environment and can be, roughly but quite reasonably, demonstrated in terms of a competitive fishing [6].

To formalize this environment, we introduced personal mindsets and their distributions for a human population in a space between two (Fig. 1) or many choices/brands (Fig. 2). In this space a specific mindset has the corresponding coordinates of its location in such a way that distances between the mindset position and the available choices/brands are relative measures of "willingness to buy" each brand (or "propensity to join or believe"). Changes in a personal mindset (stimulated by many factors including advertising activities and word of mouth) are reflected in the mindset motion between choices. Thus, in Figs. 1 and 2 the motion is presented by mindset positions for time $t = 0, 1, \ldots, 8$.

This motion is the Markovian process, which for an isolated person without motivations (no advertising, no opinion exchange, etc.) is a random walk that can be characterized by the next-step root-mean-square displacement $a$. We called the value $a$ as a measure of personal "*flexagility*". This coined term reflects causes of

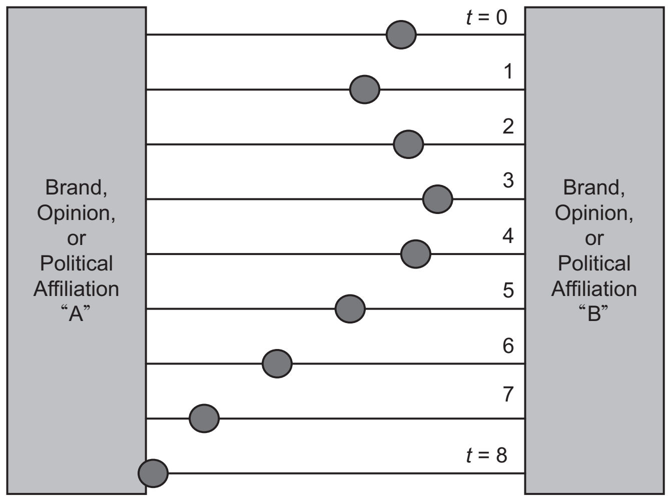


Fig. 1. Random walk between two opinions in one person's mind (floating mindsets).

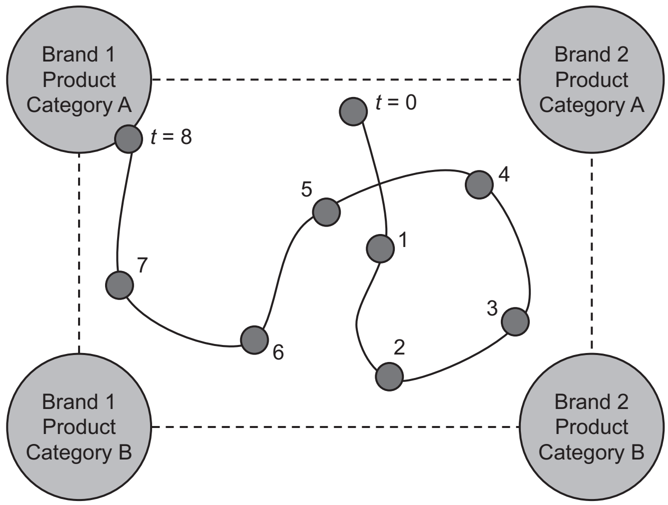


Fig. 2. Random walk between many opinions.

the ability to be displaced far enough from the previous state. A highly flexible person's mind often changes directions of the movement, an agile one moves fast in a given direction. Thus, a low flexibility combined with a high agility always provides a high displacement, while a high flexibility together with a low agility guarantees a low displacement. Two other combinations lead to different results depending on the values of these parameters. Thus, a person is flexagile if he/she changes states of mind (opinions, preferences, etc.) easily and fast in the absence of external forces. The higher the flexagility, the higher the probability that a person changes significantly his/her opinion within the next moment. The value $a$ can be different for different types of persons (*heterogeneity of the population*).

If the distance between a mindset location and a choice equals zero, then absorption occurs, which means a sale event takes place (in the figures it corresponds to $t = 8$). Changes (motion) in personal mindsets lead to the corresponding dynamics of mindset distributions for a human population. Sales (catch) dynamics is the variation of a number (or portion) of absorbed personal mindsets on a specific alternative (brand, opinion, etc.).

The sales are observable or measurable values, which for each moment of time reflects mostly the corresponding number of population having "near-brand" mindsets, i.e., being ready to buy (absorb).

However, in a long-term perspective the sales originate from the entire time-dependent population mindset distributions that usually are unknown (unobserved or unmeasured) in many parts. Indeed, the mindset distributions contribute to future sales due to the mindset motions from previous distributions, and, therefore, the knowledge about entire distribution is very important. Thus, using fishing analogy, if a fisherman knows that a huge fish shoal is approaching (the distribution is skewed to his shore, but not observed yet), he will not pack his stuff and go home. The same is true in marketing. The proposed approach can analyze and forecast the unknown distribution forms. By extra measurements (e.g. surveys), some hidden parts of a distribution can become observed, providing an additional knowledge about the distribution shape.

*Dimensionality and topology* of the mindset space depend on considering problem. If we have just two alternatives, say, the willingness to buy products of a specific brand name vs. any other brands, then the dimensionality is one, because with two choices only mindset positions along the line connecting two alternatives matter. Extra dimensions can arise from explicit multi-brand competition, competition between categories of products inside single brands, and some other real-life complexities. Special dimensions with a non-trivial topology of the space may be called "demand" and "experience".

Personal mindsets are usually subjected to many types of *motivations and influences* from other persons. These motivations and influences can be treated as forces originating from the corresponding *fields applied to personal mindsets*, which cause their trends to move in specific directions. In marketing, those motivations are reduced factors of advertising, economical situation, seasonality, influential effects ("*word-of-mouth*" information that is very hard to count in traditional-statistical approaches, but what is a hot topic in sociophysics), price, and so on.

## 3. Physical approach to the problem

Here we describe a physical technique to analyze population distribution between two or many alternatives. As it has been mentioned above, the fundamental feature of the considering system is a Markov chain probability of each unit to be within certain distance from previous state at any given time. It makes the process very similar to the Brownian motion under the action of external fields.

### 3.1. Green's functions

Green's function $G_t(\mathbf{q}_t, \mathbf{q}_0)$ is the conditional probability that the state of person's mind (or the Brownian particle) at time $t$ is placed at the point $\mathbf{q}_t$ provided the initial state was at the point $\mathbf{q}_0$. The $\mathbf{q}_t$ can be one- or multi-dimensional "*generalized coordinate*" of the state. In particular, it can include not only "real coordinate" but also, for instance, a velocity. Green's function $G_{t+1}(\mathbf{q}_{t+1}, \mathbf{q}_0)$ has the recurrence relation

$$G_{t+1}(\mathbf{q}_{t+1}, \mathbf{q}_0) = \int G_1(\mathbf{q}_{t+1}, \mathbf{q}_t) G_t(\mathbf{q}_t, \mathbf{q}_0)\, \mathrm{d}\mathbf{q}_t, \tag{1}$$

or

$$G_{t+1} = \hat{Q} G_t, \tag{2}$$

where $\hat{Q}$ is the transfer operator written as

$$\hat{Q} = \exp[-W_{t+1}(\mathbf{q})]\hat{g}. \tag{3}$$

The $W_t(\mathbf{q})$ stands for a field (in energetic temperature units for Brownian particle) applied to units in position $\mathbf{q}$ at time $t$, and $\hat{g}$ is the unit-connectivity operator which, if applied to an arbitrary function $\psi(\mathbf{q})$, can be written as

$$\hat{g}\psi(\mathbf{q}) = \int g(\mathbf{q}, \mathbf{q}')\psi(\mathbf{q}')\, \mathrm{d}\mathbf{q}'. \tag{4}$$

The function $g(\mathbf{q}, \mathbf{q}')$ describes the connection of neighboring states during time in terms of the Markov chain conditional probability in the absence of any other fields and, therefore, it is a distribution of personal flexagility. In the absence of fields (i.e., where $W_t(\mathbf{q}) \equiv 0$), Eq. (3) is reduced to $\hat{Q} = \hat{g}$ that has to be for "free" units.

For the simplest Gaussian model in the $D$-dimensional position space $\mathbf{z}$ (assuming $\mathbf{q} = \mathbf{z}$), i.e., where

$$g(\mathbf{z}, \mathbf{z}') = \left(\frac{D}{2\pi a^2}\right)^{D/2} \exp\left[-\frac{D(\mathbf{z} - \mathbf{z}')^2}{2a^2}\right] \tag{5}$$

follows to the normal distribution, in leading terms (see, for example, Ref. [7]).

$$\hat{g} \simeq 1 + \frac{a^2}{2D} \Delta_{\mathbf{z}}. \tag{6}$$

Here $\Delta_{\mathbf{z}}$ is the Laplacian operator in the position space $\mathbf{z}$, and $a^2$ is the mean-square distance (displacement) between two neighboring states during time in the absence of other fields:

$$a^2 = \int \mathbf{z}^2 g(\mathbf{z}, \mathbf{0})\, \mathrm{d}\mathbf{z}. \tag{7}$$

The operator $\hat{g}$ can be reduced from general integral form (4) to a differential one, like in (6), not only for the Gaussian case but also for many other different models (see, for instance, the case of an additional orientation memory in Refs. [8,9]).

We should emphasize that Eqs. (3)–(6) are valid for a smooth variation of the external field $W_t(\mathbf{q})$ on the time scale of each model. The reason is that for these cases we can consider the interval between two neighboring states as unperturbed by external fields and identify (2) as the Chapman–Kolmogorov equation for transitional probabilities.

In continuous limit (for a large number of time intervals $t$), we can rewrite (2) as

$$-\frac{\partial G_t}{\partial t} = (1 - \hat{Q}) G_t. \tag{8}$$

For smooth fields Eqs. (3), (6) and (8) lead to the Schrödinger-type equation:

$$\frac{\partial G_t}{\partial t} = -W_t G_t + A \cdot \Delta_{\mathbf{z}} G_t, \tag{9}$$

where $A \equiv a^2/(2D)$ is a constant.

Let us assume that the total population in the considered system is $N$. Generally speaking, $N(t)$ can be time-dependent, because of a possibility for population to grow or shrink. This value is incorporated through the normalization condition:

$$\int G_t(\mathbf{q}_t, \mathbf{q}_0)\, \mathrm{d}\mathbf{q}_t = N(t). \tag{10}$$

### *3.2. One-dimensional space for two alternatives*

Let us consider a simple case of two brands as in Fig. 1. Space between the brands $0 < z < 1$ is filled by a large number of persons. Each person is associated with a point $z$ in that space. The distances $z$ and $1 - z$ between the point and brands correspond to a person's mindset with respect to each brand, respectively. The shorter the distance, the stronger the mindset that this person has to that brand. There is a distribution of population between brands for each moment of time. Assuming an initial form of the distribution $G_{t_0}(z)$ at time $t = t_0$ we can calculate step by step corresponding distributions $G_t(z)$ for the moments $t = t_0 + 1$, $t_0 + 2$ and so on. Here, for $1D$ calculations,

$$G_t = (1 - W_t) G_{t-1} + A \cdot \frac{\mathrm{d}^2 G_{t-1}}{\mathrm{d}z^2}. \tag{11}$$

The last equation is valid for $0 < z < 1$. To calculate $G_t(z)$ at $z = 0$ and 1, we have to apply boundary conditions, which depend on considering a specific problem and are selected from a "physical meaning" of the boundaries. For instance, for non-absorption case (the only one available store is closed, or the fisherman is sleeping and does not make a catch): $G_t(0) = 0$; for sales with in-stock daily limits that are lower than demand: $\partial G_t(0)/\partial t = 0$; for uniform near-attractor area: $\partial G_t(z)/\partial z|_{z=0} = 0$, etc.

### 3.3. Motivations as applied fields

The total field $W$ is affected by many factors including economic, social, marketing, interpersonal opinion influence and others. It can be presented using different level complexity corresponding to specific objectives. We plan to describe it in detail in our next paper. Here, we only demonstrate the basic ideas behind the definition of fields, and, therefore, we present $W$ in its simplest form. In this form the total field can be presented as a sum of the following main pieces:

$$W(\mathbf{z}, t) = W_0(\mathbf{z}, t) + W_C(\mathbf{z}, t) + W_F(\mathbf{z}, t) + W_I(\mathbf{z}, t) - \overline{W}(t), \tag{12}$$

where $W_0(\mathbf{z}, t)$ and $W_C(\mathbf{z}, t)$ are the field contributions from both own and competitor's advertising activities, respectively; $W_F(\mathbf{z}, t)$ stands for the contributions from general (non-advertising) factors, like economic (for instance, Dow Jones indexes and average national prices for a product category) and social ones; $W_I(\mathbf{z}, t)$ is the influential part, which is based on interpersonal relations and opinion exchange (''word of mouth''); and $\overline{W}(t)$ is the uniform position-undependable term that added because of the following reasons. The forces, which applied to units, are determined by relative changes of fields over considering position space. Therefore, we can add or subtract a constant field (constant over the space but maybe different for each time moment $t$) without a result perturbation. The most reasonable way is to make the space-average of $W(\mathbf{z}, t)$ equals zero, i.e., $\int_V W(\mathbf{z}, t)\,\mathrm{d}\mathbf{z} = 0$, where $V = \int_V \mathrm{d}\mathbf{z}$ is the considering space volume. Then, $\overline{W}(t)$ is defined as

$$\overline{W}(t) = \frac{1}{V}\int_V [W_0(\mathbf{z}, t) + W_C(\mathbf{z}, t) + W_F(\mathbf{z}, t) + W_I(\mathbf{z}, t)]\,\mathrm{d}\mathbf{z}. \tag{13}$$

Keep considering a simple form of fields, $W_0(\mathbf{z}, t)$ and $W_C(\mathbf{z}, t)$, can be characterized directly in terms of expenses in different advertising channels. If we have $n_0$ channels of own advertising and $n_C$ channels of competitor's ones, then for $1D$ system with $\mathbf{z} = z$,

$$W_0(\mathbf{z}, t) = z^{v_0}\beta_0\left[1 + \sum_{k=1}^{n_0} B_{0k} \cdot b_{0k}(t)\right] \tag{14}$$

and

$$W_C(\mathbf{z}, t) = (1 - z)^{v_C}\beta_C\left[1 + \sum_{k=1}^{n_C} B_{Ck} \cdot b_{Ck}(t)\right], \tag{15}$$

where the powers $v_0$ and $v_C$ characterize spatial dependences of the fields with respect to the corresponding attraction centers at $z = 0$ and 1; $b_{0k}$ and $b_{Ck}$ are own and competitor's advertising expenses in channel $k$; $B_{0k}$ and $B_{Ck}$ are corresponding weight factors to be estimated; and, finally, $\beta_0$ and $\beta_C$ stand for total scaling factors. The general-factor fields can be written in a similar form:

$$W_F(\mathbf{z}, t) = |Z_F - z|^{v_F}\beta_F\left[1 + \sum_{k=1}^{n_F} B_{Fk} \cdot b_{Fk}(t)\right], \tag{16}$$

where $k$ stands for type of the corresponding factor $b_{Fk}$ and $n_F$ is the total number of accounting general factors. The scaling factors $\beta_0$, $\beta_C$ and $\beta_F$ are introduced here (unlike the usual presentation in regression models) to have the weight factors $B_{0k}$, $B_{Ck}$ and $B_{Fk}$ as the measures of relative advertising-channel effectiveness, which are more treatable than the corresponding productions $\beta_0 B_{0k}$, $\beta_C B_{Ck}$ and $\beta_F B_{Fk}$. In the last equation the attraction center is placed at $z = Z_F$, which is somehow arbitrary, but based on the reasonable assumption that general factors affect people regardless of their proximity to the ''real'' (own and competitor's) attraction centers. In general, in the people-mindset spaces with dimensionality more or equal to two, the array of points of equal distances from different attractors is interesting subject, because closeness to one node does not mechanically mean remoteness from another (for instance, a dualistic distance concept was proposed in Ref. [10]), which we are going to consider in another article.

Influential part of the fields deserves a special consideration. As we have mentioned above, there exist many attempts to analyze influential factors (word of mouth) in opinion propagation in sociophysics. Their majority is based on Ising-type models, which assume sophisticated person-to-person interaction principles. It is clear that consideration of each person individually is not always acceptable, especially for large populations. Here, we propose a different approach that is used in statistical physics to describe volume interactions between particles. This is, the so-called, self-consistent mean-field approach, where interactions (or influences here) are determined by density distributions of units over a space of positions (or opinions, as described above). In this paper, we reproduce this approach in the second (pair or unit-to-unit) virial approximation:

$$W_I(\mathbf{z}, t) = \beta_I G(z, t), \tag{17}$$

where $G(z, t)$ is the introduced above Green's function describing a unit density distribution at time $t$, and $\beta_I$ is the scaling factor.

## 4. Links to modern statistical techniques

Among consequences of the basic principles of the proposed mediaphysical approach, we can recognize many concepts postulated in modern statistics. On one hand, this makes the approach naturally related with modern statistics. On the other hand, the links between the mediaphysical approach and some of the concepts make clearer not only the origination of the concepts but also their positioning in a whole picture of analyzed phenomena.

Here we demonstrate the straightforward links of the proposed approach with (a) lag concepts and their modifications including trend analysis, Box–Jenkins models or adstock ones in marketing [11,12], and (b) random-coefficients mixed models (or yield analysis) [13]. Both concepts, in fact, are generalizations of large classes of models in time series analysis and regression and very important for multiple applications. To start, let us to remind briefly the forms of basic equations in each of the concepts.

### *4.1. Traditional statistics*

*The regression with lags concept* usually declares the following dependencies of a target variable (say, own sales) $S(t)$ on a set of $m$ factors:

$$\mathbf{b}(t) \equiv \{b_1(t), b_2(t), \ldots, b_m(t)\} \tag{18}$$

for each moment $t$:

$$S(t) = \Theta(t) + \sum_{i=0}^{i_0} [\mathbf{C}_i \cdot \mathbf{b}(t-i)], \tag{19}$$

where $\mathbf{C}_i \equiv \{C_{1i}, C_{2i}, \ldots, C_{mi}\}$ are coefficients that are different for each lag $i = 0, 1, \ldots, i_0$ and $\Theta(t)$ is the time-dependable baseline. The coefficients $\mathbf{C}_i$ can be interrelated and obeyed a specific function of $i$ and a couple of parameters (for example, two-parameter gamma function for adstock models in Ref. [12]). The baseline can include an explicit trend:

$$\Theta(t) = \Theta(t_0) + (t - t_0) \left.\frac{\partial \Theta(t)}{\partial t}\right|_{t_0}. \tag{20}$$

The set of factors $\mathbf{b}(t)$ can include own and competitor's advertising as in Eqs. (14) and (15), and general factors as in Eq. (16). However, in the traditional-statistical approaches the influential factors (word of mouth) of Eq. (17) cannot be included. Thus, we can write $m = n_0 + n_C + n_F$.

*The random-coefficients mixed models* are intended to estimate coefficients (yields) on each data point and can be presented as

$$S(t) = \Theta(t) + [\mathbf{C}^0 + \delta \mathbf{C}(t)] \mathbf{b}(t), \tag{21}$$

if no lag effects, or in a form similar to Eq. (20) with lag effects but with fluctuating coefficients for each lag

$$\mathbf{C}_i(t) = \mathbf{C}_i^0 + \delta\mathbf{C}_i(t), \tag{22}$$

i.e.,

$$S(t) = \Theta(t) + \sum_{i=0}^{i_0}\{[\mathbf{C}_i^0 + \delta\mathbf{C}_i(t)]\mathbf{b}(t-i)\}. \tag{23}$$

Here, $\mathbf{C}_i^0$ is a constant value for each lag $i$ and $\delta\mathbf{C}_i(t)$ is the fluctuating part of the factor's coefficient (i.e., it depends on $t$). The last equation is a generalization that includes both concepts of regression with lags and random-coefficient mixed models.

### *4.2. Mediaphysics vs. traditional statistics*

The links of mediaphysics with the above concepts can be established from Eq. (11) and the relation between the target (sales) value $S(t)$ (below in this paragraph we will write it as $S_t$) and Green's function at the absorption ($z = 0$) point $(G_t)_0 \equiv G(z,t)|_{z=0}$:

$$S_t = \lambda \cdot (G_t)_0, \tag{24}$$

where the scaling factor $\lambda$ is the same for both own and competitor's sales. Then,

$$S_t = (1 - W_t^0)S_{t-1} + A\lambda \cdot (\Delta_{\mathbf{z}} G_{t-1})_0, \tag{25}$$

where $W_t^0 \equiv W(z,t)|_{z=0}$ is the total field at the absorption point $z = 0$ at time $t$. Using the recurrence relation (25) extra $i_0$ times, we can write

$$S_t = S_{t-1-i_0}\prod_{i=0}^{i_0}(1 - W_{t-i}^0) + A\lambda\sum_{i=0}^{i_0}\left\{\left[\prod_{j=1}^{i}(1 - W_{t-j}^0)\right](\Delta_{\mathbf{z}} G_{t-1-i})_0\right\}. \tag{26}$$

Here, $S_{t_0} \equiv S_{t-1-i_0}$ and $G_{t_0}(z) \equiv G_{t-1-i_0}(z)$ can be considered as initial sales and population density distribution, respectively, at $t_0 \equiv t - 1 - i_0$.

The fields that are composed of factors $\mathbf{b}(t)$ in Eq. (18) are explicitly included in Eqs. (12)–(16), but with no word-of-mouth effects of Eq. (17). Therefore, we can write

$$W_t^0 = U_t^0 + \widetilde{W}_t^0, \tag{27}$$

where the term that included all the factors (18) can be written as

$$U_t^0 = -\mathbf{B}^* \cdot \mathbf{b}(t) \tag{28}$$

with the reduced coefficient vector $\mathbf{B}^*$; and other fields are included in $\widetilde{W}_t^0$. For smooth fields the production

$$\prod_{i=0}^{i_0}(1 - W_{t-i}^0) \simeq \left(1 - \sum_{i=0}^{i_0}\frac{U_{t-i}^0}{1 - \widetilde{W}_{t-i}^0}\right)\prod_{i=0}^{i_0}(1 - \widetilde{W}_{t-i}^0) \tag{29}$$

and in leading terms Eq. (26) can be written as

$$S_t = \Theta(t) + \Theta(t)\sum_{i=0}^{i_0}\frac{\mathbf{B}^* \cdot \mathbf{b}(t-i)}{1 - \widetilde{W}_{t-i}^0} + A\lambda\sum_{i=0}^{i_0}(\Delta_{\mathbf{z}} G_{t-1-i})_0 \tag{30}$$

or

$$S_t = \Theta(t) + \sum_{i=0}^{i_0}\{[\mathbf{C}_i^0 + \delta\mathbf{C}_i(t)]\mathbf{b}(t-i)\} + A\lambda\sum_{i=0}^{i_0}(\Delta_{\mathbf{z}} G_{t-1-i})_0, \tag{31}$$

where

$$\Theta(t) \simeq S_{t_0}\left(1-\sum_{j=0}^{i_0}\widetilde{W}^{0}_{t-j}\right) \tag{32}$$

$$\mathbf{C}_i^0 \simeq S_{t_0}\mathbf{B}^* \tag{33}$$

and

$$\delta\mathbf{C}_i(t) \simeq S_{t_0}\mathbf{B}^*\left(\widetilde{W}^{0}_{t-i}-\sum_{j=0}^{i_0}\widetilde{W}^{0}_{t-j}\right). \tag{34}$$

The baseline $\Theta(t)$ in Eq. (32) can be also easily presented in a form with trends as in Eq. (20).

First two terms in Eq. (31) are in agreement with the generalized equation (23) for lags and random-coefficients mixed models. Mathematical expressions for coefficients (32)–(34) are approximations for the purpose of links demonstration only. Simultaneously, here we show the main difference between the mediaphysical approach and common statistical techniques: it is the last term in Eq. (31) which is absent in Eq. (23). This term is basically what this mediaphysical approach is about: the population connectivity in a distribution, the system ‘‘inertia’’, and the complex factors like word-of-mouth opinion exchange. The last phenomena, while being very important, are not analyzed by traditional statistics. Observed results (sales/catches) are not only direct responses to some external factors even with lags and coefficient fluctuations, but also consequences of time-dependent structures of the complex organized systems. The differences in results from traditional statistical and mediaphysical approaches are demonstrated below in the case study.

## 5. Model implementation and interpretation

### 5.1. General notes

Now we have a mathematical model to deal with. The starting point is an initial population density distribution, which has to be in agreement with market share or own-to-competitor ratio of average catches (sales or voters) from first available time points. Some parameters of the initial distribution can be fitted later to adjust system dynamics.

Then, we calculate system behavior during time in terms of Green's function, using the equations described above, which for 1$D$ system are (11)–(17) with the normalization (10). In Fig. 3 we present an example of

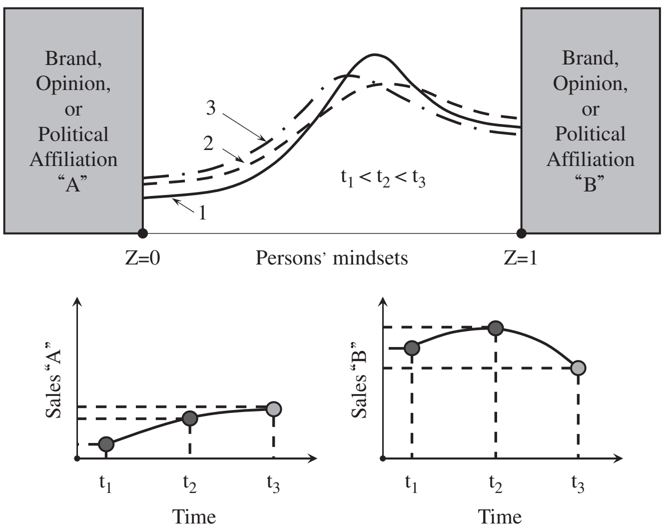


Fig. 3. Population-distribution dynamics and sales.

population-distribution dynamics following this methodology and the corresponding time-dependent absorption (sales) on the own and competitor's attraction centers at $z = 0$ and 1.

The dollar amount of sales (or number of buyers) is proportional to Green's function at the absorption points or areas. In $1D$ case, own and competitor's sales are $S_0(t) = \lambda \cdot G(z,t)|_{z=0}$ and $S_C(t) = \lambda \cdot G(z,t)|_{z=1}$, respectively, with the single scaling factor $\lambda$.

In this approach, the majority of effects can be analyzed by numeric calculations and solutions of the equations. However, we can provide analytical estimations for some results: for instance, short-term effects (depending on time scales it can be next day, week or month) of advertising for a specific channel $k$ (e.g. local radio or national TV channel). Thus, a relative increase in sale dollars per advertising dollar can be written as

$$\frac{1}{S_0}\frac{\partial S_0}{\partial b_{0k}} = \frac{\beta_0}{v_0 + 1} B_{0k} \tag{35}$$

and

$$\frac{1}{S_C}\frac{\partial S_C}{\partial b_{Ck}} = \frac{\beta_C}{v_C + 1} B_{Ck} \tag{36}$$

for own and competitor's results, respectively. Then, relative short-term effectiveness of different channels (e.g. $k_1 =$"Local Radio" vs. $k_2 =$"National TV") for own and competitor's brands can be calculated as $(B_{0k_1}/B_{0k_2})$ and $(B_{Ck_1}/B_{Ck_2})$. Moreover, the short-term competition effectiveness for channel $k$ is

$$E_k = \frac{\beta_0 B_{0k}(v_C + 1)}{\beta_C B_{Ck}(v_0 + 1)}. \tag{37}$$

However, let us emphasize once more that last estimations in Eqs. (35)–(37) are valid for short-term effects. Long-term effects can be much more complicated and have to be analyzed under numeric computations.

Parameters of the model, which are not defined from the very beginning, have to be determined from the best fitting of model. Then, the adjusted parameters are to be used for forecasting, optimization of decisions, and for what-if analysis. To estimate these parameters we used an optimization procedure developed specially for this methodology (a kind of genetic-type algorithm with stochastic components). However, it is possible to use any reliable optimization procedure dealing with non-linear behavior with many local extremes and completely non-analytical definition.

### 5.2. *Case of a real media efficiency problem*

This approach has been applied to a real media efficiency problem for a large company. It demonstrated very good results despite the low initial correlations of factors and target variable.

To emphasize the basic advantages of the approach in our example, we consider a simple 1D model with two generalized competitive brands for a specific type of product: the first is our "own" brand, the second all the competitor's brands together. Business details cannot be disclosed because of confidentiality.

Observed variables are time-dependent sales: (1) our own company's sales and (2) total sales for all competitors. Real sales are represented in monthly increments by circles in Fig. 4. Because competitors' data combine many companies, they are usually significantly higher than for own company.

The information included in the filled circles (for 43 months, left of the thick solid vertical line) was used historically to create a model and to estimate model parameters, including strengths of the external and internal fields and motivations. Then, using the model with the estimated parameters, we forecasted both our own and our competitors' sales simultaneously for the next 12 months (on the right from thick line) and compared them with the real sales for those months (open circles in the figure). The model results are presented by the solid curves for both historical and forecasted areas in both our own and our competitors' dependencies.

There are several noteworthy aspects of this case study. First, the mediaphysics approach examines both our own and our competitors' sales simultaneously in the frameworks of a single complex model, which is quite unusual for traditional statistics.

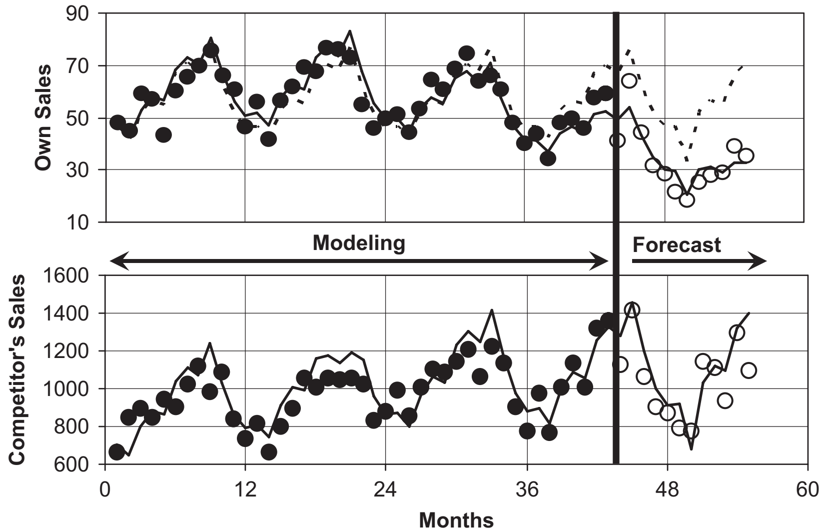


Fig. 4. Real-data model and forecast for both own and competitor's sales simultaneously. Solid curve is for mediaphysics approach and dashed one for traditional regression.

Second, the study's forecasting accuracy is an improvement over the traditional-statistical approaches discussed above. Moreover, it is significantly better if the forecasting area does not follow a simplistic reaction as in the demonstrated case of Fig. 4, where our own sales dropped significantly in the forecasting months. The mediaphysical approach was able to catch the corresponding long-term reasons of the drop. The traditional regression analysis, however, leads to much direr forecasting, with high overestimation of the sales (see dashed curve in own-sales part of the figure). The random-coefficient mixed model, discussed above, leads to a higher correlation between model and real data than simple regression in the historical part, but almost cannot improve the regression accuracy in forecasting.

Long-term or even infinitely long-term effects, which can be accounted for in mediaphysics, are v ery important not only to increase forecasting accuracy, but also to understand what is the reason behind a phenomena and how it can be treated by a company in short- and long-term perspectives. In some cases even small changes in advertising activities can lead to blockbuster or catastrophic results due to some secondary effects of "word of mouth" and mechanisms of opinion propagation in specific population distributions.

Not as abrupt but still important are the effects of ceasing our own advertising activity for a short period of time, which is demonstrated in Fig. 5 for the same model as in Fig. 4. This what-if analysis "switched off" our own advertising completely for a period marked by gray in the figure after initial 12 months. Then, following the gray period, advertising was resumed to a level as in the original scheme. The curve in Fig. 5 demonstrates the dynamics of the corresponding percentage of sales decrease over time. Dropping sales do not stop instantly after the gray period, and the sales never resume on the initial level after a long period of time, as would be the case using a traditional-statistics regression approach. Instead, sales dynamics caused by the gray period are slowly stabilized on a lower level.

Another example of the what-if analysis using the current approach and the model shown in Fig. 4: assuming a 10% uniform increase in advertising spending for competitors' brands during one year with its resumption on the original-scheme level for the next years, we can analyze appropriate responses from our own brand company: no increase in own advertising spending, the same 10% increase or a higher increase (see Figs. 6 and 7). In the figures the second year (gray area) cover a period when competitors' spending increases 10%. The curves describe the percentage of the corresponding sales changes of our own company for each month separately in Fig. 6 and in a cumulative form in Fig. 7. Different curves associate with different responses in our own brand ad spending: (a) no changes, (b) 10% uniform increase, and (c) 20% increase for each month starting from the beginning of the gray area and to the end of diagram.

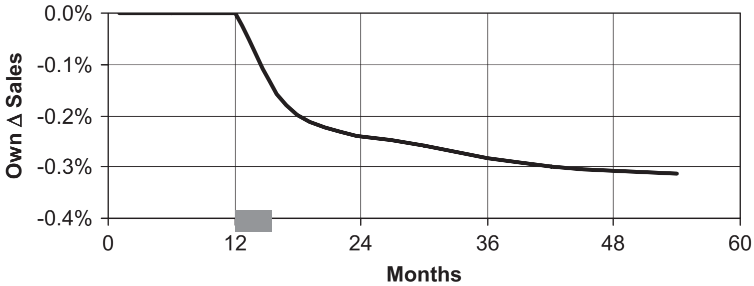


Fig. 5. Real-data long-term sales effects of short-term (gray area) elimination of advertising spending.

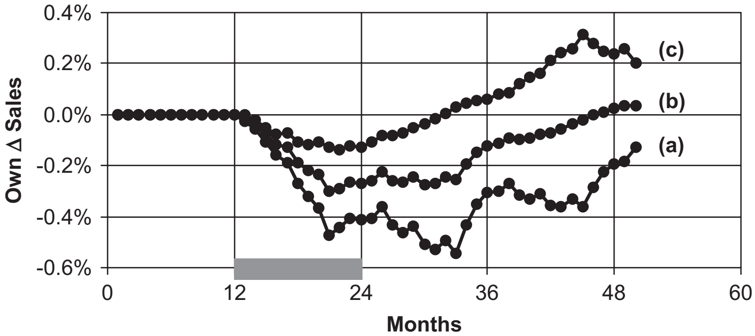


Fig. 6. What-if analysis: monthly sales effects of one year (gray area) 10% uniform increase of competitor's ad spending with three possible responses from the own brand starting from the same year (gray area) and up to the end of diagram: (a) no changes in the own ad spending, (b) 10% uniform increase, and (c) 20% uniform increase.

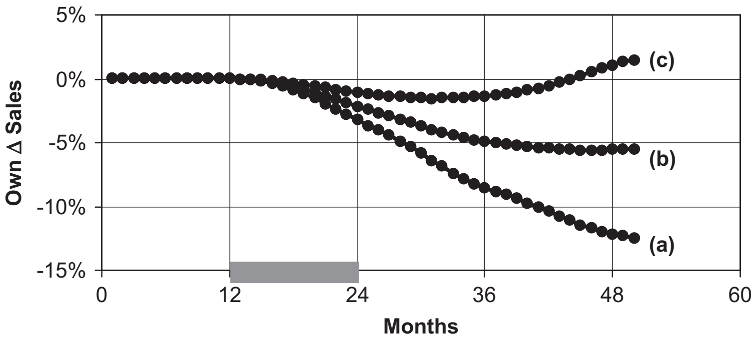


Fig. 7. The same as in the previous figure, but for cumulative effects.

## 6. Conclusions

We developed a new *mediaphysical* approach to analyze a statistical dynamics of real-life systems, composed by many partly-unobserved units (usually human population), where mass communications (both internal and external) form their behavior. To apply this method to marketing, social life and politics, we introduced a concept of "*persons*' *mindsets*" *space* with distributed attractive and repulsive forces. We considered and formalized the concepts of the corresponding space dimensionalities, the distances in this space, and the relations between states of an individual mind and the distribution of all individuals. It is shown that, under reasonable assumptions, the system dynamics may be considered as a specific Markovian process, described in terms of Green's functions and Schrödinger-type equation. We showed links of the mediaphysical approach and such important concepts of traditional statistics, as effects lasting in time (Box–Jenkins models, adstock in marketing), and random-coefficient mixed models (or yield analysis), what makes mediaphysics naturally related with modern statistics. Simultaneously, it allows answering some questions, which are usually not or poorly addressed in traditional statistical and/or simulational sociophysical models:

1. A strong distinction between (a) data having mass structure with a distribution in some space, where only selected points can be observed (like marathon runners with TV camera maintained on start and finish lines only), and (b) data with only observed units (like single runner with a personal camera) was grounded. Respectively, for the first data type the mediaphysical approach is preferable. Traditionally, in statistics this distinction was not considered.

2. The proposed method organically allows considering both *short- and long-term effects* of marketing and propaganda in a consistence form, avoiding subjective definitions of lag or similar structures.

3. The technique allows formulating, checking and applying hypothesis about unobserved population *homogeneity or heterogeneity*, replacing traditional-statistical variables like "percent of educated women over time" by distributions of personal behaviors ("flexagility" or mindset displacements). Thus, it strongly distinguishes variables, describing process and those describing population itself, which usually in statistics is bulked together.

4. The mediaphysical approach allows accounting for person-to-person interactions (*influence, word of mouth*) inside the population (the leading topic of sociophysics, which is not usually addressed in statistics) in a more general way through self-consistent fields, that provides a unique opportunity to consider completely different components of social behavior together within one model, even when the number of factors is big (unlike traditional sociophysics simulation with very limited number of factors).

5. It predicts *future system dynamics* from its history in such a way that *system connectivity and inertia* are naturally taken into account. As a result, very important aspects of the forecasts (like prediction of hidden bumps of unobserved distributions) may be derived and used in practice.

6. For many problems the method allows replacing a simulation by analytical calculations or combining two approaches in a reasonable format.

Here, we just introduced basic ideas and methodology that we put in the proposed approach. These concepts should be developed much deeper. Among next-steps priorities we may list more detailed enhancements of the multifield scenarios and influential fields, non-trivial topologies of the considering space, better optimization procedures, and many other problems. An important topic could be a what-if procedure for different scenarios and assumptions.

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## Appendix A. Sociophysics—a battle for existence

Sociophysics has acquired growing popularity for the last decades, but still is "... at its childhood" [4]. Its subject and terminology, and even its short history have been topics of confusions and misinterpretations.

### *A.1. On the sociophysics history and terminology*

Historically, different analogies between social life and physics (while not exactly application of physics to social life) were spread over the centuries, from Empedocles [5] to Condorcet at the earlier times. On a formal level, most likely, the first application of a physical model (random walk) to a social phenomenon was the famous now, but largely unknown in its day, study of financial speculation by L. Bachelier in 1900 [14].

Some works on the physics of both economics and sociology sporadically appeared during the last century up to the early 1980s without pronouncing a specific field/branch of science (see, as examples, Refs. [15–17] and other works cited in Refs. [18–20]). The term "social physics" was initially proposed by August Compte, but became popular after publishing the book "Sur l'homme et le developpement de ses facultés, essai d'une physique sociale'' by A. Quetelet (1835) [21], which became famous not only among scientists, but even for its period's general public. A. Compte, being vastly disappointed by the fact that such a good term was ''stolen'', had to invent another one, ''sociology'', what made his name immortal (en.wikipedia.org/wiki/Auguste_-Compte), while ''physique sociale'' lost its popularity very fast and was rediscovered one and a half century later. Unfortunately, authors were not yet able to find original sources of that puzzling ''crime story'', but it is very characteristically that ''sociology'', ''social physics'', and ''statistics'' (see below) were in fact considered almost as synonyms at the time of inception of all of them.

During the last two to three decades, the growing number of the ''social physics'' studies was put under the flag of sociophysics—a term reestablished in 1982 for modern times by S. Galam, Y. Gefen and Y. Shapir in Ref. [18]. A much larger number of economical physics studies was defined as ''econophysics'', as proposed by H.E. Stanley in 1995. Econophysics is mostly dedicated to applications of statistical physics to stock-options pricing and portfolio optimization [22,23]. There is also smaller number of studies in the more traditional economical areas [24]. In our opinion, if social sciences include economics then, generally speaking, sociophysics should include econophysics.

It is worth noting that ''physique sociale'' has not only linguistic and historic association with sociophysics, but also a more serious link than it may first seem. Being an astronomer and mathematician, Quetelet was the first person who applied mathematics (theory of the recently discovered normal distribution, used for measuring of astronomical errors) to social life. This is why many scientists consider him the founder of quantitative sociology. Strictly speaking, he did not directly use any physical methods. But his interpretation of ''social physics'' as a science about finding solid rules in an ocean of random individual social fluctuations was of enormous importance for statistics and sociology, directly affecting such thinkers as F. Galton, K. Marx, E. Durkheim, and, probably, C. Darwin, among others. Solidness of statistical averages was exactly the reason why he called it ''physics,'' associating the deterministic physics of his time with that new discovered amazing phenomenon in social life. In his words, social physics will ''... present laws quite as admirable as the mechanics of inanimate objects'' [25].

But Quetelet's influence was more than that. His work, very likely, inspired J.C. Maxwell to create a kinetic gas theory [26], i.e., the boundary between physics and ''social physics'' at that time did not seem too firm, though ''a flow'' was in opposite direction than it is today.

The rush he created around normal distribution in social life was dubbed in the 1920s ''Quetelismus'' and it was ultimately denied for the lack of universalism (though normal distribution presumption still occupies a huge place in statistics). That rush perfectly matches the current rush with power law distributions (there are tens of works, see just [22,24,27]), which feeds a big part of econophysics and sociophysics.

The logic behind those two rush waves is methodologically the same: in both cases some physical models of data generation were proposed, not only empirical observation: (1) Quetelet assumed a unique nature of social constants-averages, in Newtonian sense, with random fluctuations due to unobserved reasons; (2) modern authors prefer agent-based models, or criticality and some other effects (see Ref. [27]).

The fact that Quetelet's work did not spark an immediate interest to apply physics to social life (rather his social studies stimulated physics) does not mean that he has no prominent place in as yet unwritten history of sociophysics. It could be fair to say, that Quetelet considered social science, somehow naively from the perspective of our time, as a sort of physics, whereas modern scientists are talking about using physics in the study of social life. Nevertheless, he tightly and consciously linked those two fields for the first time, making very general notion of A. Compte much more concrete.

Remarkably, some arguments today in defense of the right of sociophysics for existence repeat Quetelet's points about new science almost literally. In 1835 Quetelet wrote (his italic): ''... *the greater the number of individuals observed, the more do individual peculiarities, whether physical or moral, become effaced, and leave in a prominent point of view the general facts, by virtue of which society exists and is preserved''* (see Ref. [21, p. 6]). In 1982 Galam, Gefen and Shapir emphasized (authors' italic): ''There is, however, one aspect of human behavior which is directly related to physics; namely, *collective behavior*. In physics this term refers to situations where the behavior of each small constituent of a large system is correlated with the behavior of others, and all components of the system lose their individual character'' [18]. In 2004 Stauffer stated: ''Whether I smoke, drink vodka in the morning, and eat steaks every evening influences my date of death, and neither employer nor health insurance know about it. Nevertheless, by averaging over millions of people, these personal details cancel out ... Thus humans and atoms may be described by the same method, if we look at averages.'' [5]

But if Quetelet defended what he called ''physique sociale'' (and what others called ''statistics''), modern authors defend what they called ''sociophysics,'' appealing to the same line of arguments that validated the same statistics for 150 years or more (recall also law of big numbers, etc. of that kind). This does not diminish in any way those cited and other authors' contribution to the new field, it just states that boundaries between statistics, sociophysics, and ''physique sociale'' remain fuzzy if one does not apply more precise definitions.

### *A.2. On the sociophysics subject*

And, indeed, some uncertainty exists about Sociophysics subject. In Ref. [3], the only modern book (1993 and second edition in 2005) we know under that title, the sociophysics has a super-universal meaning and embraces both nature and social life within an entire paradigm (*universal definition*). The latest reviews [5,28], do not provide exact definitions, but actually describe a much more narrow field, limited mainly (but not only) by computer simulations of social phenomena based on (supposedly, but not always in practice) physical principles (*simulation definition*). In this definition, sociophysics becomes almost undistinguishable from social simulation (see Journal of Artificial Societies and Social Simulation, issuing 9 years, while no journal for sociophysics exists yet). The corresponding analytical approaches are less developed. However, many interesting analytical studies were published during last few decades (see, for instance, Refs. [15,19,29,30]). As stated above, we may tentatively define sociophysics as a science about the application of physics in social sphere (*broad definition*).

A big plus of the simulation approach is its simplicity for formulation of a simulated scenario, and hence the number of possible models is limited only by researcher's curiosity. Many interesting results were already obtained from this approach, though it has some intrinsic problems. Simulations have significant limits in numbers of objects and involved factors. And these restrictions are, indeed, observed in a majority of publications that makes them far from measured reality while capturing some directional and qualitative observations [31]. For the history of sociophysics simulation (at least from about 100 papers published last five years we reviewed) just a few examples of real applications were presented. Typically, sociophysics simulation models do not use real data to estimate models' parameters. Such freedom from reality allows, on one hand, checking exotic hypotheses, but on the other hand, leaves too much room for arbitrary decisions. Those aspects are considered in Ref. [32], where another possible branch, based exclusively on agent-based modeling, was discussed.

One of the most cited applications of the sociophysics simulations is a modeling of election results in Brazil, which was called ''... at present the strongest validation of the Sznajd Model'' since its introduction in 2000 [33,34]. The qualitative agreement with the election results was achieved even without many crucial real-life factors. However, it is doubtful that real-life election results depend only on a positive interaction of people; and that advertising (partly simulated in Refs. [35,36]), negative word-of-mouth effects, and the political programs of candidates [37] (factors, which were not considered originally) do not affect the results. Also worth mentioning to the practical application of sociophysics is Galam's approach to minority opinion spreading [38]. The model predicts a possibility of initial-minority winning if a reform is debated for a long time, a tendency, which can be confirmed by some European events.

The problem of physical-model applicability to social life inevitably raises questions about relations of this model and statistics, which is the only tool to detect ''factors'' and ''causes'' in sociology. Thus, the statistical causal theory [39] applies very sophisticated methods to separate ''causation vs. correlation'' (still used only for narrow class of phenomena). This problem in physics usually stands in a very different manner (see, for instance, Ref. [3]) and usually does not stand in simulations at all, because everything there is under an experimenter's control. The danger than is that impressive results in an idealized simulated world may be even directionally wrong in real application.

For instance, if only interactions between people can explain election results, it can be easily imagined that if one would apply only advertising he could get the same results. It may happen because advertising is a cause of neighbors convincing (people are convinced not because they are neighbors, listening to each other, but because they also watched the same TV show yesterday). This effect is widely known as multicollinearity in statistics, but not in statistical physics. So, there is a methodological gap between these two ways of looking at the same social phenomena. The current simulational sociophysics does not fill the gap, while the proposed mediaphysics can do it for a wide class of problems.

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